HIGH SPEED DIGITAL FILTERING
BY
CONVOLUTION APPROXIMATION

E. B. WEIS, JR., MD

FEBRUARY 1967

AEROSPACE MEDICAL RESEARCH LABORATORIES
AEROSPACE MEDICAL DIVISION
AIR FORCE SYSTEMS COMMAND
WRIGHT-PATTERSON AIR FORCE BASE, OHIO

NOTICES

When US Government drawings, specifications, or other data are used for any purpose other than a definitely related Government procurement operation, the Government thereby incurs no responsibility nor any obligation whatsoever, and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data, is not to be regarded by implication or otherwise, as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use, or sell any patented invention that may in any way be related thereto.

Requests for copies of this report should be directed to either of the addressees listed below, as applicable:

Federal Government agencies and their contractors registered with Defense Documentation Center (DDC):

DDC
Cameron Station
Alexandria, Virginia 22314

Non-DDC users (stock quantities are available for sale from):

Chief, Storage and Dissemination Section
Clearinghouse for Federal Scientific & Technical Information (CFSTI)
Sills Building
5285 Port Royal Road
Springfield, Virginia 22151

Organizations and individuals receiving reports via the Aerospace Medical Research Laboratories' automatic mailing lists should submit the addressograph plate stamp on the report envelope or refer to the code number when corresponding about change of address or cancellation.

Do not return this copy. Retain or destroy.

HIGH SPEED DIGITAL FILTERING
BY
CONVOLUTION APPROXIMATION

E. B. WEIS, JR., MD

Foreword

This work was conducted in support of Project 7233, "Biological Information Handling Systems and Their Functional Analogs," Task 723301, "Biological Mechanisms for Signal Analysis," and Task 723305, "Theory of Information Handling."

The period of conduct of this work was January 1964 to June 1966.

This technical report has been reviewed and is approved.

J. W. HEIM, PhD
Technical Director
Biomedical Laboratory
Aerospace Medical Research Laboratories

Abstract

This report discusses a technique for digital filtering by convolution approximation which is an acceptable compromise between accuracy and speed. This technique is applicable where high accuracy is not necessary and where a digital computer with elaborate processing capability is not available.

Most requirements for digital filtering can be developed in terms of the approximations suggested in this report. By casting the required filter in a form that is most amenable to numerical computation, the accuracy of approximation is maximized. If power-of-two accuracy is insufficient there is a continuous trade-off between accuracy and speed which involves range division and more bits in the approximation.

SECTION I.
Introduction

It is frequently desirable to be able to accomplish high-speed digital filtering. The need arises for preprocessing of data for online calculations, studies of nonstationary signals where the non-stationarity is primarily due to a certain frequency spectrum and as an alternative to detailed harmonic analysis (to specify three examples) (ref 1, ref 3).

A great deal of effort has been expended to make digital filtering practical in computation time, cost, and accuracy of the technique of digital filtering. There are two general techniques known to the author. These are the convolution or Ormsby filters (ref 1) and the weighted average filters which include the recursive filters.

This discussion is not intended to review the entire problem of digital filtering, but rather to explore a technique for implementing the convolution technique in a manner that is relatively accurate and very fast in computation time for certain digital computers.

SECTION II.

The Theory of Convolution Filters

The output of a linear filter (not necessarily realizable) can be given as follows:

$$F^*(w) = H(w) \; F(w) \tag{1}$$

where $F^*(w)$ is the Fourier Transform of the filter output

$F(w)$ is the Fourier Transform of the filter input

$H(w)$ is the Fourier Transform of the filter impulse response

Equation 1 has a time domain representation as follows:

$$f^*(\gamma) = \int_{-\infty}^{\infty} h(t) \; f(t-\gamma) \; dt \tag{2}$$

where $f^*(\gamma)$ is the filter output

$f(t)$ is the filter input

$h(t)$ is the filter impulse response

A numerical approximation of equation (2), using the trapezoidal rule for integration, can be written as follows:

$$f^*(\gamma_j) = \sum_i [h(t_i) \; f(t_i-\gamma_j) + h(t_{i+1}) \; f(t_{i+1}-\gamma_j)] \left[\frac{t_{i+1}-t_i}{2} \right] \tag{3}$$

If the t_i are equally spaced:

$$t_{i+1} - t_i = \Delta t \tag{4}$$

and therefore:

$$f^*(\gamma_j) = \frac{\Delta t}{2} \sum_i [h(t_i) \; f(t_i-\gamma_j) + h(t_{i+1}) \; f(t_{i+1}-\gamma_j)] \tag{5}$$

In general $h(t_i)$ is nonzero almost everywhere. However, let us restrict the considerations to the following case:

$$h(t_i) \approx 0 \qquad I_1 \Delta t = T_{min} < t_i < T_{max} = I_2 \Delta t \tag{6}$$

Therefore,

$$f^*(\gamma_j) = \frac{\Delta t}{2} \sum_{I_1}^{I_2} [h(t_i) \; f(t_i-\gamma_j) + h(t_{i+1}) \; f(t_{i+1}-\gamma_j)] \tag{7}$$

2

This development is accurate for an arbitrary $f(t)$ but in fact, practical considerations limit the application to cases where:

$$\int_{-\infty}^{\infty} f(t) \; dt = 0 \qquad (8)$$

If the $f(t)$ does not fulfill this condition then write $f(t)$ as follows:

$$f(t) = f_1(t) + A \qquad (9)$$

where $\int_{-\infty}^{\infty} f_1(t) \; dt = 0$

and $\qquad \int_{-\infty}^{\infty} f(t) = A$

Then equation 7 can be rewritten as follows:

$$f^*(\gamma_j) = \frac{\Delta t}{2} \sum_{I_1}^{I_2} [h(t_i) \; (f_1(t_i - \gamma_j) + A) + h(t_{i+1}) \; (f_1(t_{i+1} - \gamma_j) + A)] \qquad (10)$$

$$f^*(\gamma_j) = A \; \frac{\Delta t}{2} \sum_{I_1}^{I_2} [h(t_i) + h(t_{i+1})]$$

$$+ \frac{\Delta t}{2} \sum_{I_1}^{I_2} [h(t_i) \; f_1(t_i - \gamma_j) + h(t_{i+1}) \; f_1(t_{i+1} - \gamma_j)] \qquad (11)$$

or

$$f^*(\gamma_j) = f_1^*(\gamma_j) + A \int_{I_1}^{I_2} h(t) \; dt \qquad (12)$$

This shows that one can correct for a nonzero mean process by subtracting the mean in the beginning and adding the mean times the integral of the impulse response of the filter at the end.

The previous development will suffice for low-pass operations and the following for high-pass operations.

$$f_h^*(t) = f(t) - f_1^*(t) \qquad (13)$$

where $f_h^*(t)$ is a desired high-pass operation

$f_1^*(t)$ is the complementary low-pass operation

$f(t)$ is the unfiltered data.

Therefore, a high-pass operation can be accomplished by subtracting from the data the result of passing the data through a low-pass filter whose transfer characteristic is the complement (in terms of the band-pass) of the desired high-pass filter.

Implementation of the convolution filter in this form on a digital computer involves a multiplication and a sum for each t_i for each γ_j. The speed of such a process in certain computers is often a severe limitation of the acceptability of this filtering technique.

SECTION III.

Approximations to the Filter

The primary disadvantage of the previous form lies in the time required for multiplication. However, it is obvious that if the values of the filter impulse response were integral powers of two, slow multiply steps could be replaced with fast shift steps in the digital computer. Therefore, the question arises; how well does the ideal filter characteristic compare with the filter that has values which are the nearest (in some sense) integral power of two?

It is clearly possible to expand $h(t)$ in a binary series to any accuracy desired.

$$h(t_m) = u(t_m) \sum_j a_j 2^j \qquad j = 0, 1, 2 \cdots \qquad (14)$$

where $u(t_m) = \begin{matrix} 1 \\ -1 \end{matrix} \Big\} \begin{matrix} h(t_m) \geq 0 \\ h(t_m) < 0 \end{matrix}$

$a_j = \begin{matrix} 0 \\ 1 \end{matrix} \Big\}$ otherwise for each j corresponding to the presence of the jth power of two

This is, of course, the way a number is represented in a digital computer and so implementation of a multiplication by expansion in a binary series is equivalent to binary multiplication with N bits of accuracy. Our discussion concerns implementation with only a few bits of accuracy.

Another point is that it is possible to logically divide the range of $h(t)$ in implementing the convolution so that one might expand the filter weights as follows:

$$h(t_m) = u(t_m) \sum_j a_j 2^j \qquad h(t_m) \leq S_1 \qquad (15)$$

$$= u(t_m) (S_1 + \sum_j a_j 2^j) \qquad S_1 < h(t_m) \leq S_2$$

$$\vdots$$

$$= u(t_m) (S_n + \sum_j a_j 2^j) \qquad S_n < h(t_m) \leq S_{n+1}$$

Where the $\{S_i\}$ are arbitrary level divisions.

For the sake of simplicity consider nearest in the following sense. Choose n_i to be the integer value such that:

$$| h(t_i) - u(t_i) a_i 2^{n_i} | \quad \text{is a minimum}$$

An integral square error criterion has not been considered. It may be that the approximation could be improved thereby.

Using the power-of-two approximation one has:

$$f_1{}^*(\gamma_j) = \frac{\Delta t}{2} \sum_{I_1}^{I_2} [u(t_i) a_i 2^{n_i} f_1(t_i - \gamma_j) + u(t_{i+1}) a_{i+1} 2^{n_{i+1}} f_1(t_{i+1} - \gamma_j)] \qquad (17)$$

Using the logical level division and the power-of-two approximation one has:

$$f_1{}^*(\gamma_j) = \frac{\Delta t}{2} \sum_{I_1}^{I_2} [\, u(t_i)\ (S_n + a_i 2^{n_i})\ f_1(t_i - \gamma_j) + u(t_{i+1})\ (S_n + a_{i+1} 2^{n_{i+1}})\ f_1(t_{i+1} - \gamma_j) \,] \tag{18}$$

or

$$f_1{}^*(\gamma_j) = \frac{\Delta t}{2}\ S_n \sum_{I_1}^{I_2} [\, u(t_i)\ f_1(t_i - \gamma_j) + u(t_{i+1})\ f_1(t_{i+1} - \gamma_j)$$

$$+ \frac{\Delta t}{2} \sum_{I_1}^{I_2} [\, u(t_i) a_i 2^{n_i}\ f_1(t_i - \gamma_j) + u(t_{i+1})\ a_{i+1} 2^{n_{i+1}}\ f_1(t_{i+1} - \gamma_j) \,] \tag{19}$$

where

$$S_n < h(t_i) \leqq S_{n+1}$$

A computer program has been written for the IBM 7044-7094 direct coupled system to generate the approximation for a filter of the following type:

$$h(t) = \frac{w_c}{\pi}\ \frac{\sin(w_c t)}{(w_c t)} \tag{20}$$

where $w_c = 2\pi f_c$

f_c is the cutoff frequency

The Fourier Transform for this function is:

$$H(w) = 1 \qquad -w_c \leqq w \leqq w_c$$
$$\ = 0 \qquad \text{otherwise}$$

The time domain graphs for the function and its approximation are illustrated in figure 1.

Using an existing computer program, SYSTRAN (ref 2), the Fourier transforms of the function shown in equation 20 and its power-of-two approximation were calculated. The results are illustrated in figures 2A and 2B.

The filter accuracy for the division of the range of the filter values has not been evaluated.

There is a further consideration in decreasing the time involved in convolving filters of the $(\sin x)/x$ type. This function is even, therefore, the number of shift operations can be reduced by half. Further, one can reduce the number of shift operations again since the function crosses some bit levels more than once. The number of shift operations can also be reduced by using subtract instructions (instead of add) when the filter weight is negative.

$$f_1{}^*(\gamma_j) = \frac{\Delta t}{2} \sum_{n_i} a_i 2^{n_i} \sum_{2^{n_i}} u(t_i)\ f_1(t_i - \gamma_j) \tag{21}$$

The internal sum in equation 21 is for all values of $f_1(t_i - \gamma_j)$ which are to be multiplied by a particular 2^{n_i}. The external sum is over all possible 2^{n_i}.

6

SECTION IV.
The Accuracy of the Approximate Filter

The power-of-two approximation has been found to show an average error (relative to the exact filter) of about 5% in the passband and rolloff range (over a limited frequency range). In the stop-band range one finds that the power-of-two filter admits somewhat more signal energy than the exact filter. However, the approximate filter is still as much as 60 dB down.

The rate of rolloff is primarily determined by the length of the convolution (i.e., I_1 and I_2). Using 2¼ cycles of $(\sin x)/x$, the rolloff approaches 140 dB/octave.

One can generate an exact expression for the error of the approximate filter, but it is always in the form of a series. Therefore, short of evaluating the series for an actual case, the error cannot be given a detailed numerical bound.

The appendix shows the results of decreasing the length of the convolution and increasing the interval between samples in the convolution. This material is presented in the form of graphs on which the pertinent parameters are listed (see figures 1 through 6B).

Appendix

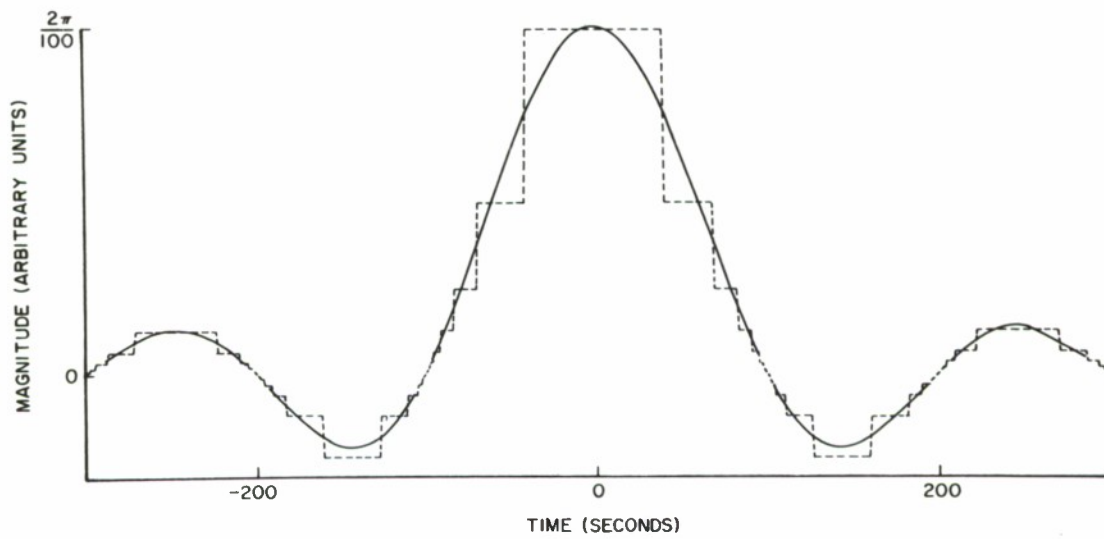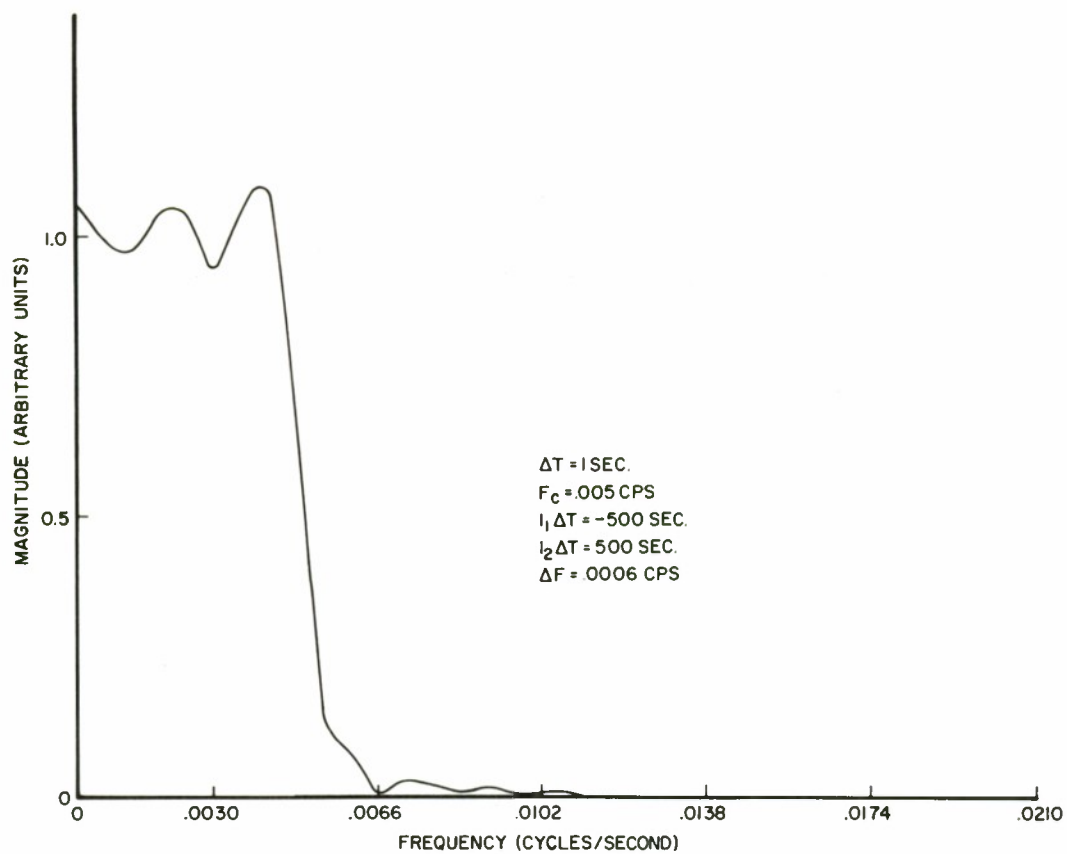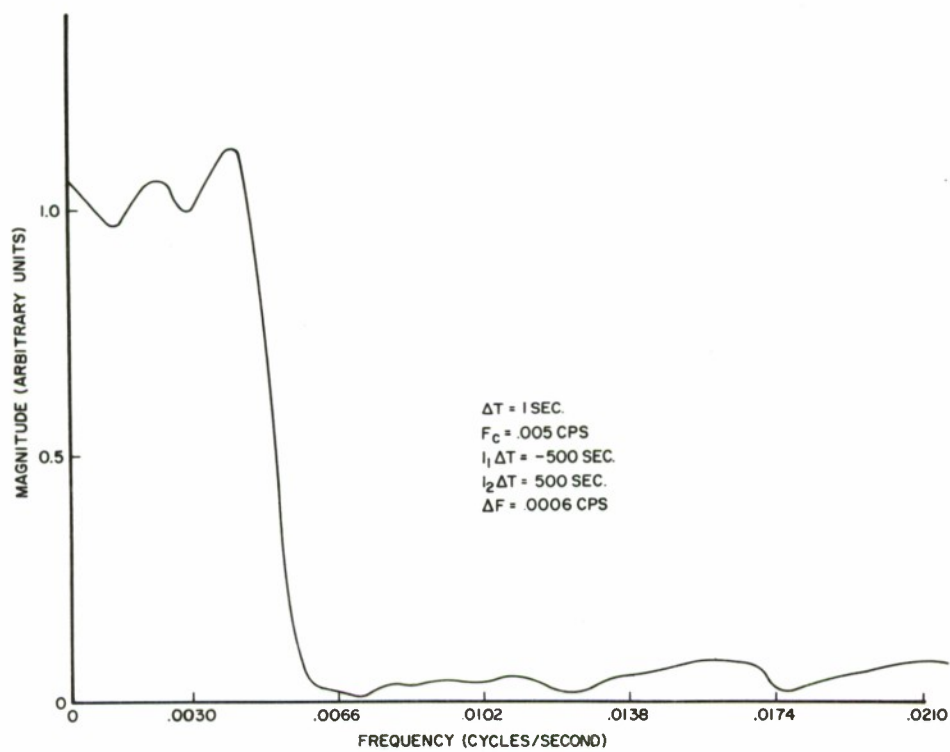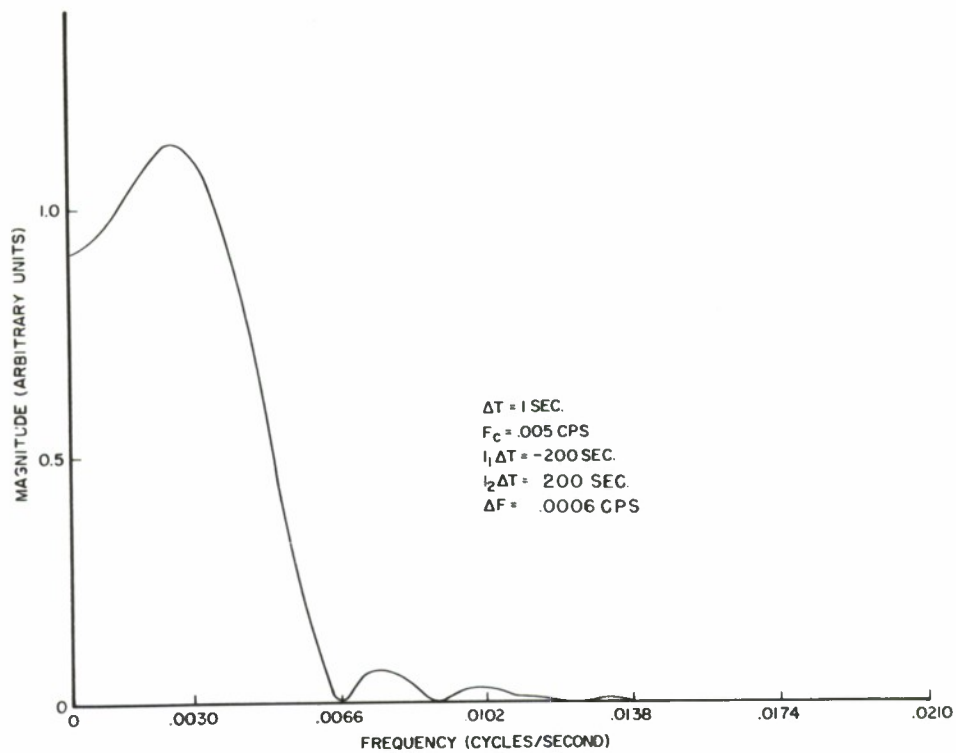## Examples of the Spectra of a Filter and its Power-of-Two Approximation

FIGURE 1. **Filter Function and Approximation**

FIGURE 2A.   **Fourier Spectrum of the Filter**

$$\Delta T = 1\,\text{SEC.}$$
$$F_c = .005\,\text{CPS}$$
$$I_1 \Delta T = -500\,\text{SEC.}$$
$$I_2 \Delta T = 500\,\text{SEC.}$$
$$\Delta F = .0006\,\text{CPS}$$

FIGURE 2B.   **Fourier Spectrum of the Filter Approximation**

$$\Delta T = 1\,\text{SEC.}$$
$$F_c = .005\,\text{CPS}$$
$$I_1 \Delta T = -500\,\text{SEC.}$$
$$I_2 \Delta T = 500\,\text{SEC.}$$
$$\Delta F = .0006\,\text{CPS}$$

$\Delta T = I$ SEC.
$F_c = .005$ CPS
$I_1 \Delta T = -200$ SEC.
$I_2 \Delta T = 200$ SEC.
$\Delta F = .0006$ CPS

FIGURE 3A.   **Fourier Spectrum of the Filter**

$\Delta T = I$ SEC.
$F_c = .005$ CPS
$I_1 \Delta T = -200$ SEC.
$I_2 \Delta T = 200$ SEC.
$\Delta F = .0006$ CPS
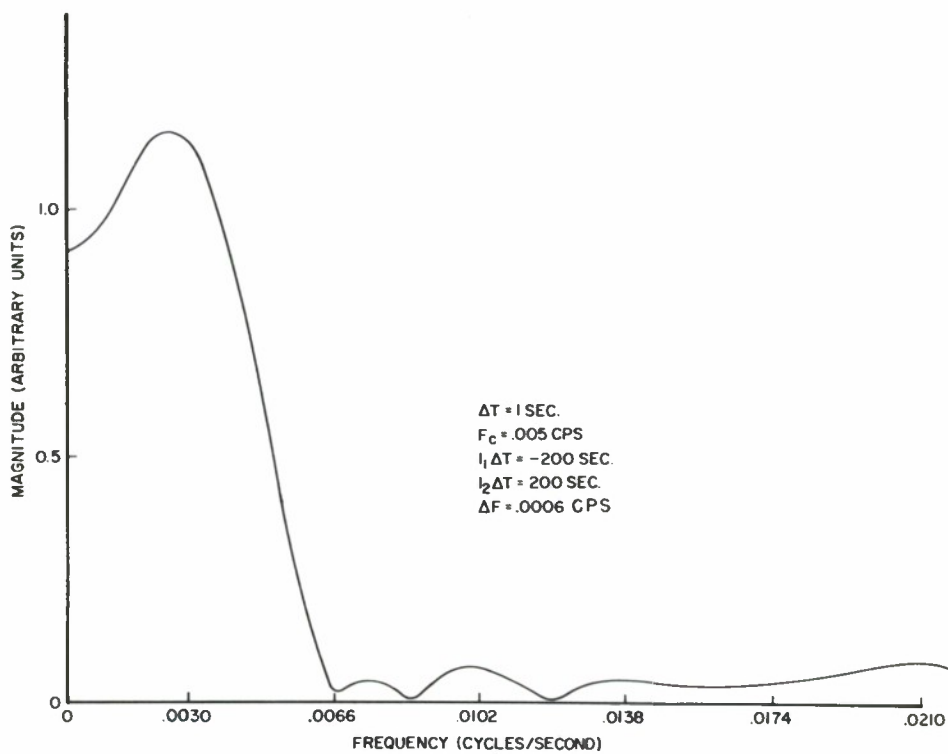
FIGURE 3B.   **Fourier Spectrum of the Filter Approximation**

ΔT = I SEC.
F_c = .005 CPS
I_1 ΔT = −100 SEC.
I_2 ΔT = 100 SEC.
ΔF = .0006 CPS
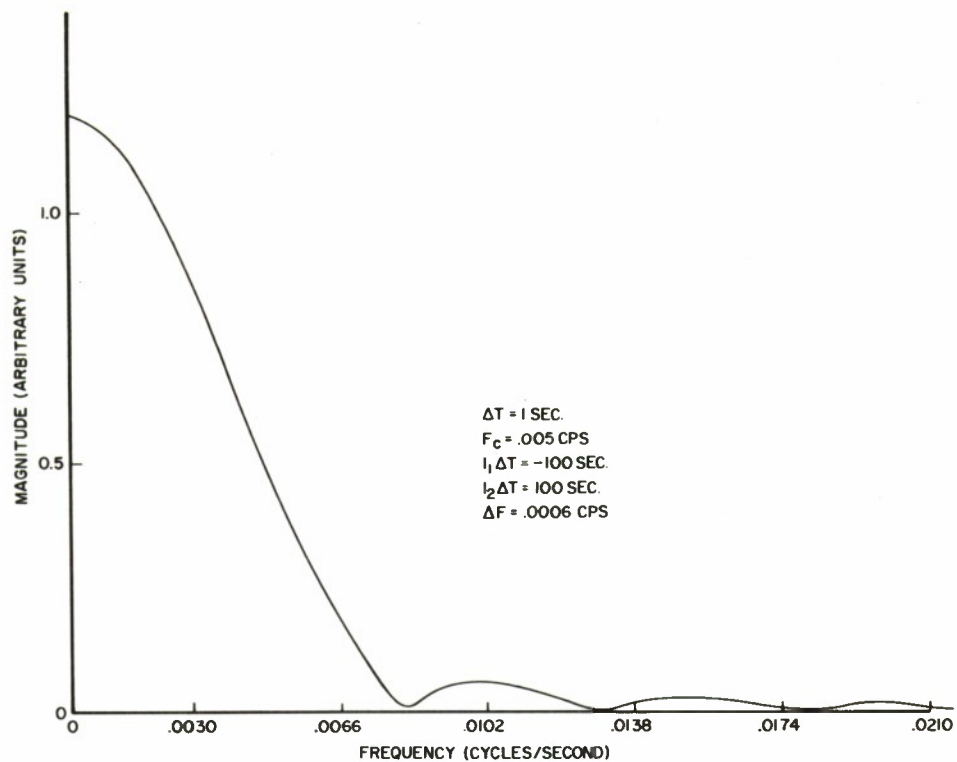
FIGURE 4A. Fourier Spectrum of the Filter

ΔT = I SEC.
F_c = .005 CPS
I_1 ΔT = −100 SEC.
I_2 ΔT = 100 SEC.
ΔF = .0006 CPS
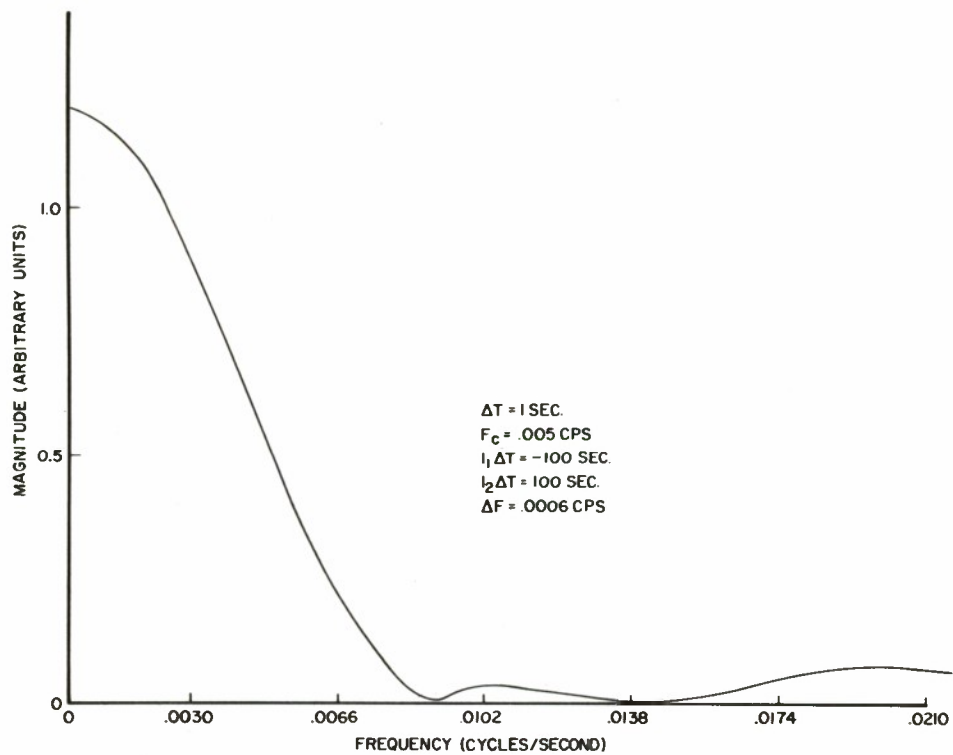
FIGURE 4B. Fourier Spectrum of the Filter Approximation
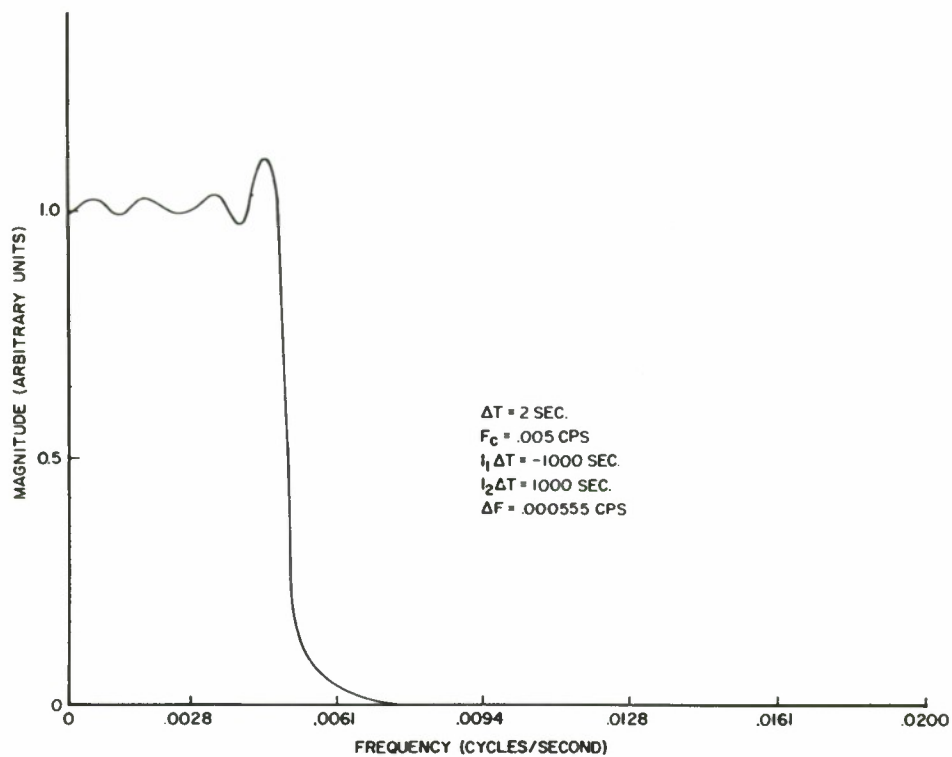
FIGURE 5A.   **Fourier Spectrum of the Filter**

$\Delta T = 2$ SEC.
$F_c = .005$ CPS
$I_1 \Delta T = -1000$ SEC.
$I_2 \Delta T = 1000$ SEC.
$\Delta F = .000555$ CPS
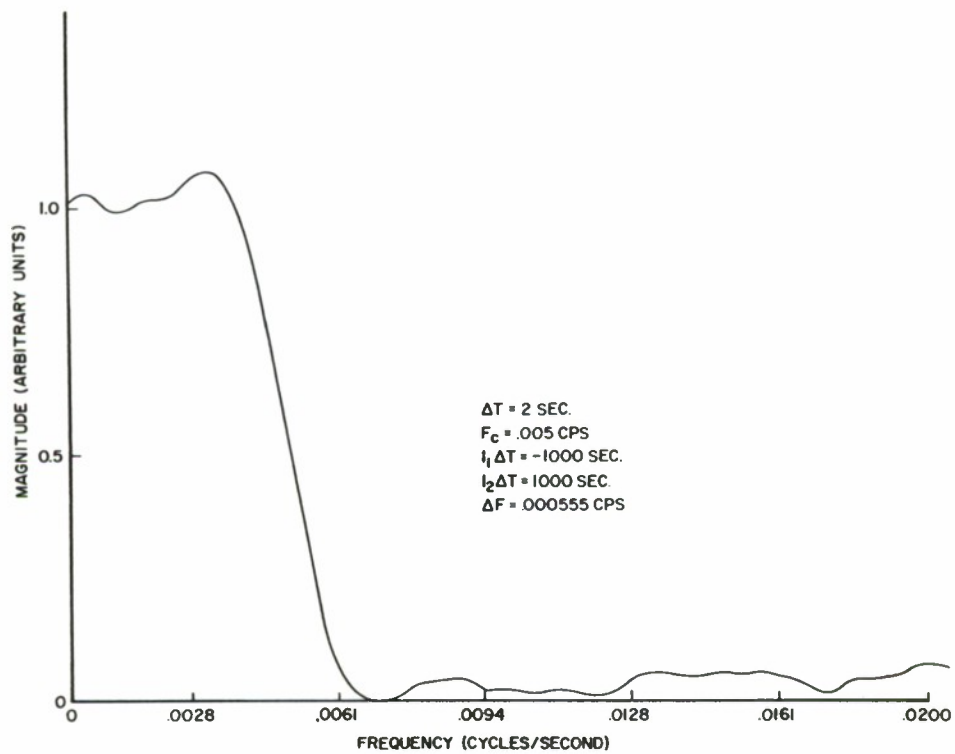
FIGURE 5B.   **Fourier Spectrum of the Filter Approximation**

$\Delta T = 2$ SEC.
$F_c = .005$ CPS
$I_1 \Delta T = -1000$ SEC.
$I_2 \Delta T = 1000$ SEC.
$\Delta F = .000555$ CPS
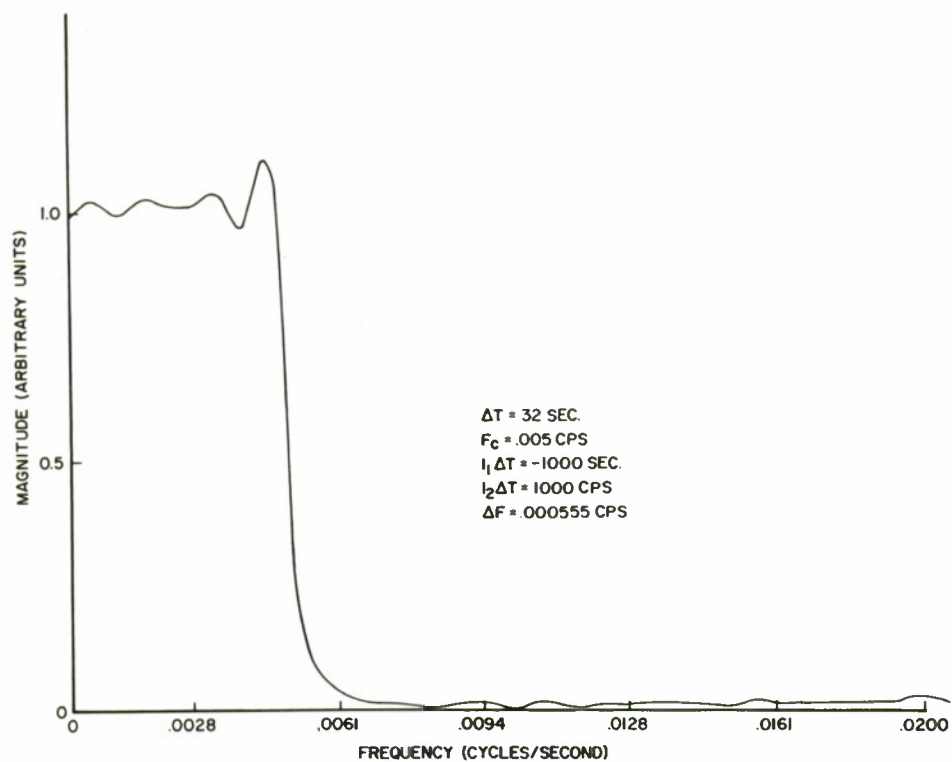
FIGURE 6A.   Fourier Spectrum of the Filter
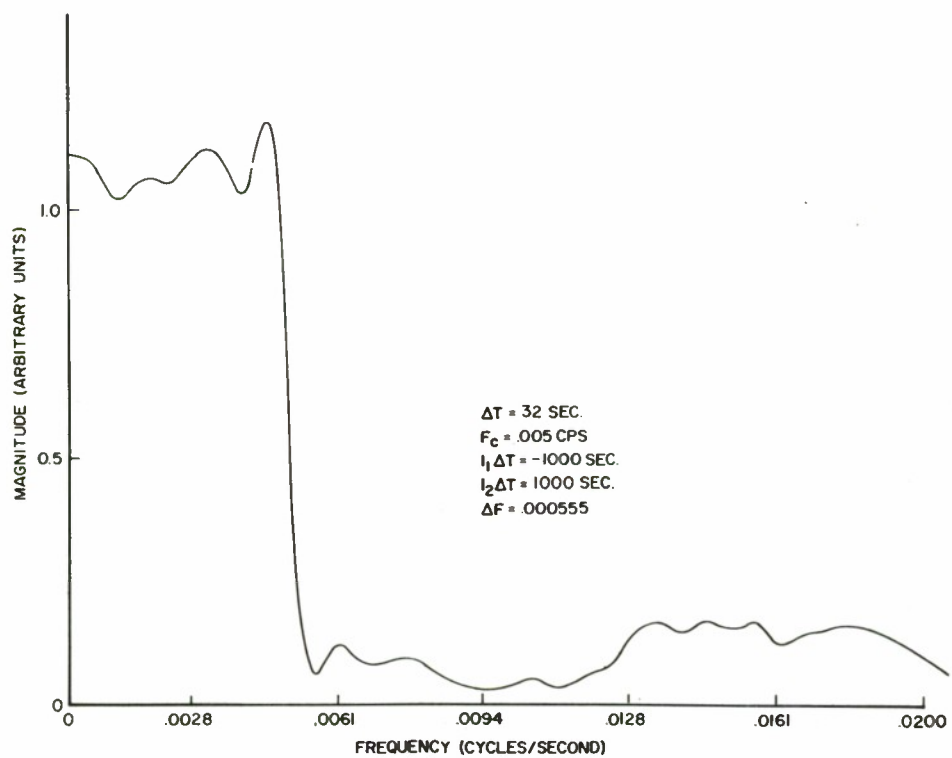
FIGURE 6B.   Fourier Spectrum of the Filter Approximation

14

References

1. Ormsby, J. F. A., "Design of Numerical Filters with Applications to Missile Data Processing," *J.A.C.M.*, Vol. 8, No. 4, pp. 440-466, 1961.

2. Lajeunesse, D. J., E. B. Weis, Jr., and T. J. Hogan, Jr., *SYSTRAN (Systems Analysis Translator): A Digital Computer Program,* AMRL-TR-65-133 (AD 624 468), Aerospace Medical Research Laboratories, Wright-Patterson Air Force Base, Ohio, July 1965.

3. Stauffer, W. M., J. C. Dill, and R. W. Stacy, "Real-Time Numerical Filtering of Physiological Signals," *IEEE Trans. on Biomedical Eng.*, Vol. BME-12, Nos. 3 & 4, pages 195-197.

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)	2a. REPORT SECURITY CLASSIFICATION
Aerospace Medical Research Laboratories	UNCLASSIFIED
Aerospace Medical Div., Air Force Systems Command	2b. GROUP
Wright-Patterson Air Force Base, Ohio 45433	N/A

3. REPORT TITLE

HIGH SPEED DIGITAL FILTERING BY CONFOLUTION APPROXIMATION

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Final Report – January 1964–June 1966

5. AUTHOR(S) (Last name, first name, initial)

Weis, E. B., Jr.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
February 1967	15	3

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT NO. 7233	AMRL-TR-66-140
c. Task No. 723301 & 723305	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	

10. AVAILABILITY/LIMITATION NOTICES

Distribution of this document is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Aerospace Medical Research Laboratories Aerospace Medical Div., Air Force Systems Command, Wright-Patterson AFB, Ohio

13. ABSTRACT

This report discusses a technique for digital filtering by convolution approximation which is an acceptable compromise between accuracy and speed. This technique is applicable where high accuracy is not necessary and where a digital computer with elaborate processing capability is not available. Most requirements for digital filtering can be developed in terms of the approximations suggested in this report. By casting the required filter in a form that is most amenable to numerical computation, the accuracy of approximation is maximized. If power-of-two accuracy is insufficient there is a continuous tradeoff between accuracy and speed which involves range division and more bits in the approximation.

DD FORM 1 JAN 64 **1473**

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Digital filtering						
Convolution technique						
Data processing systems						
Ormsby filters						
Linear filter						
Fourier analysis						

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization (*corporate author*) issuing the report.

2a. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7a. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8a. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers (*either by the originator or by the sponsor*), also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

  (1) "Qualified requesters may obtain copies of this report from DDC."

  (2) "Foreign announcement and dissemination of this report by DDC is not authorized."

  (3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through

        _____ ."

  (4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through

        _____ ."

  (5) "All distribution of this report is controlled. Qualified DDC users shall request through

        _____ ."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring (*paying for*) the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as (TS), (S), (C), or (U).

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, rules, and weights is optional.